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1994 J. Phys. A: Math. Gen. 27 3149

(http://iopscience.iop.org/0305-4470/27/9/027)

J. Phys. A: Math. Gen. 27 (1994) 3149–3158. Printed in the UK

Continuous vacua in bilinear soliton equations

J Hietarinta†§, A Ramani† and B Grammaticos‡

† CPT, Ecole Polytechnique, 91128 Palaiseau, France

‡ LPN, Université Paris VII, Tour 24-14, 5ème étage, 2 Place Jussieu, 75251 Paris, France

Received 1 November 1993, in final form 7 February 1994

Abstract. We discuss the freedom in the background field (vacuum) on top of which the solitons are built. If the Hirota bilinear form of a soliton equation is given by $A(D_x)G \cdot F = 0$, $B(D_x)(F \cdot F - G \cdot G) = 0$, where both A and B are even polynomials in their variables, then there can be a continuum of vacua, parametrized by a vacuum angle ϕ. The ramifications of this freedom for the construction of one- and two-soliton solutions are discussed. We find, for example, that once the angle ϕ is fixed and we choose $u = \tan^{-1} G/F$ as the physical quantity, then there are four different solitons (or kinks) connecting the vacuum angles $\pm\phi$, $\pm\phi \pm \pi/2$ (where π is the defined modulo). The most interesting result is the existence of a 'ghost' soliton; goes over to the vacuum but interacts with 'normal' solitons by giving them a finite phase shift.

1. Introduction

The existence of multisoliton solutions has been considered as a very strong indication of the complete integrability of nonlinear evolution equations. The main tool for this was developed by Hirota (1971), who proposed a bilinear formalism based on the observation that, if expressed in the correct variables, soliton solutions are just polynomials in exponentials [1]. Indeed, if we start from the solitary wave solution

$$u = \frac{k^2/2}{[\cosh((kx - pt)/2)]^2} \qquad k^3 - p = 0 \tag{1}$$

of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2}$$

and then introduce a new dependent variable F by $u = 2\partial_x^2 \log F$, we simply obtain $F = 1 + \mathrm{e}^{kx-pt}$.

The KdV equation itself can be cast in the bilinear form using the new variable F. From the potential form of (2), which is $v_t - {v_x}^2 + v_{xxx} = 0$, where $v_x = u$, we find that

$$FF_t + FF_{xxxx} - 4F_xF_{xxx} + 3{F_{xx}}^2 = 0 \tag{3}$$

and using Hirota's D operators defined through

$$D_x^a D_t^b \ldots F \cdot G = (\partial_x - \partial_x')^a(\partial_t - \partial_t')^b \ldots F(x, t, \ldots)G(x', t', \ldots)|_{x'=x,t'=t,\ldots} \tag{4}$$

§ On leave of absence from: Department of Physics, University of Turku, FIN-20500 Turku, Finland (email: hietarin@utu.fi).

we can write (3) as

$$(D_x^4 + D_x D_t) F \cdot F = 0. \tag{5}$$

Expressed in this new dependent variable F, the multisoliton solutions of the KdV equation can be systematically constructed (with $\eta = kx - pt$):

(i) zero soliton (vacuum): $F = 1$;

(ii) one soliton: $F = 1 + e^{\eta}$;

(iii) two solitons: $F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}$ with $A_{12} = k_1 - k_2/(k_1 + k_2)$;

(iv) three solitons: $F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}$,

etc.

We remark that the three-soliton solution does not contain any new free parameters, i.e. once A_{ij} is fixed at the two-soliton level the parameters of the three-(and higher) soliton solutions are fixed. Thus, for these higher-order solutions to exist, it is necessary for certain compatibility conditions to be satisfied.

Returning to (5), we emphasize that there exists a solution corresponding to a 'vacuum', i.e. the absence of solitons. One can see that $F = \text{constant} \neq 0$ is a solution which (due to bilinearity) can be normalized to $F = 1$. Thus, the vacuum in the case of KdV is uniquely defined.

However, the vacuum solution is not always unique. In the following, we will show that for certain multicomponent bilinear equations there is a continuum of vacua which cannot be scaled away. In this case, the solitons turn out to interpolate between different vacua and there is one instance where the soliton is a 'ghost': it is only visible in interactions with other solitons. This multiplicity of vacua also puts extra constraints on the existence of soliton solutions and, thus, on the integrability of the equation.

2. Continuous vacua for two-component bilinear equations

Bilinear equations can be classified according to the number of dependent functions one has to introduce in order to be able to write them in Hirota's form. Following the analysis in [2], we concentrate on two generic classes of two-component bilinear equations which can be written as

$$\begin{cases} A(D_x) G \cdot F = 0 \\ B(D_x) G \cdot F = 0 \end{cases} \tag{6}$$

where A and B are, respectively, the even and odd polynomials in the D operator (the modified-KdV family) and

$$\begin{cases} A(D_x)(F \cdot F - G \cdot G) = 0 \\ B(D_x) F \cdot G = 0 \end{cases} \tag{7}$$

where A and B are both even (the sine–Gordon family).

The even–odd case is 'standard' in that only the usual vacua exist. We can put $F = s$, $G = c$, but due to bilinearity we can scale this to $F = 1$, $G = 1$ or $F = 1$, $G = 0$ or $F = 0$, $G = 1$ so that no continuous parameters remain.

In this paper we focus on the even–even case (7). The sine–Gordon equation is the simplest and best known member of this family, corresponding to $A = D_x D_t$, $B = D_x D_t - 1$

[3]. If we use the standard method of nonlinearization and put $F = e^v \cos u$, $G = e^v \sin u$ in (7), with these A and B, we get

$$\begin{cases} u_{xt} = [-v_{xt} + \frac{1}{2}] \tan(2u) \\ v_{xt} = u_{xt} \tan(2u) \end{cases} \tag{8}$$

and, eliminating the field v, we get the sine–Gordon equation for $U = 4u$:

$$U_{xt} = \sin U. \tag{9}$$

Let us now return to the general case and look for the zero-soliton solution. Two types of standard vacua can readily be found. If $A(0) = 0$, $B(0) \neq 0$ then $G = 0$, $F = 1$ and $F = 0$, $G = 1$ are the only possible vacua. Also, if $A(0) \neq 0$, $B(0) = 0$ we must take $F = G = 1$ or $F = -G = 1$. Note that if neither $A(0)$ nor $B(0)$ vanish then there are no simple zero-soliton solutions at all.

A novel possibility exists if both $A(0)$ and $B(0)$ vanish in (7). Then if F and G are both constants, one gets a vacuum solution, whatever their ratio. We can use a symmetric parametrization of the continuous vacuum solution through

$$F = \cos\phi \qquad G = \sin\phi \tag{10}$$

where ϕ is a free parameter.

The construction of a one-soliton solution on top of one of the standard vacua is straightforward and has been discussed, for example, in [2]. The question we will now address is whether we can find one-soliton solutions on top of the new vacuum (10). Let us start with the ansatz

$$F = c + Ce^{\eta} \qquad G = s + Se^{\eta} \tag{11}$$

where $c = \cos\phi$, $s = \sin\phi$ and $\eta = kx - pt + \text{constant}$. Substituting into (7), we obtain the following conditions:

$$B(\boldsymbol{p})(sC + cS) = 0 \qquad A(\boldsymbol{p})(cC - sS) = 0 \tag{12}$$

where $\boldsymbol{p} = (k, p)$. Three cases can be distinguished, corresponding to three different types of solitons:

(α) $A(\boldsymbol{p}) = 0$ and $B(\boldsymbol{p}) \neq 0$, then $C = sM$, $S = cM$ for some constant M. This constant can be absorbed into the constant in η so that

$$F = c(1 + e^{\eta}) \qquad G = s(1 - e^{\eta}). \tag{13}$$

(β) $A(\boldsymbol{p}) \neq 0$ and $B(\boldsymbol{p}) = 0$, then

$$F = c + se^{\eta} \qquad G = s + ce^{\eta}. \tag{14}$$

(γ) $A(\boldsymbol{p}) = 0$ and $B(\boldsymbol{p}) = 0$, then there is, at this level, no condition on C and S and the one-soliton solution seems to be arbitrary.

A typical situation where a γ-type soliton exists is when the Hirota polynomials A and B have a common factor U: $A = UV$, $B = UW$ and thus the dispersion manifold common to both A and B contains $U(\boldsymbol{p}) = 0$. Moreover, in many cases (in particular for known integrable even–even systems), U is linear, i.e. $U(\boldsymbol{p}) = \boldsymbol{\lambda}\cdot\boldsymbol{p}$. This will have further implications for the two-soliton solution as we shall see in the next section.

3. Constraints from the existence of two-soliton solutions

In addition to the continuous parametrization of the vacuum, there is a further discrete ambiguity. We can observe this from (13), (14) when we look at the physical quantity $u = \tan^{-1} G/F$: it has different (constant) values at $\eta \to \pm\infty$, which suggests different vacua there. This will be discussed further in section 5, but before doing that let us see what the formal requirement for the existence of a two-soliton solution yields.

Let us start by combining solutions of the types α and β. First, for $\alpha + \alpha$ (which means that the dispersion relations are $A(p_1) = A(p_2) = 0$ for both of the individual solitons), the ansatz for F and G can be written as

$$\begin{aligned} F &= c + ce^{\eta_1} + ce^{\eta_2} + Ke^{\eta_1+\eta_2} \\ G &= s - se^{\eta_1} - se^{\eta_2} + Le^{\eta_1+\eta_2} \end{aligned} \tag{15}$$

and then we obtain, from (7), $K = cM$, $L = sM$ with

$$M = -\frac{A(p_1 - p_2)}{A(p_1 + p_2)} = \frac{B(p_1 - p_2)}{B(p_1 + p_2)}. \tag{16}$$

This implies, in particular, a condition for the existence of two-soliton solutions. On the manifold $A(p_1) = A(p_2) = 0$ we must have

$$A(p_1 + p_2)B(p_1 - p_2) + A(p_1 - p_2)B(p_1 + p_2) = 0. \tag{17}$$

This condition reflects the fact that, generically, even–even bilinear equations do not have two-soliton solutions.

The $\beta + \beta$ solution leads to analagous results

$$\begin{aligned} F &= c + se^{\eta_1} + se^{\eta_2} - cMe^{\eta_1+\eta_2} \\ G &= s + ce^{\eta_1} + ce^{\eta_2} - sMe^{\eta_1+\eta_2} \end{aligned} \tag{18}$$

with M given by (16) and (17) as the existence condition (but now, however, on a different dispersion manifold, namely $B(p_1) = B(p_2) = 0$).

The case $\alpha + \beta$ is treated in a similar way. We start from

$$\begin{aligned} F &= c + ce^{\eta_1} + se^{\eta_2} + Ke^{\eta_1+\eta_2} \\ G &= s - se^{\eta_1} + ce^{\eta_2} + Le^{\eta_1+\eta_2} \end{aligned} \tag{19}$$

on $A(p_1) = B(p_2) = 0$ (or $B(p_1) = A(p_2) = 0$) and find $K = -sM$, $L = cM$ with (16) and (17) as the compatibility condition (on yet a different dispersion manifold). Thus for standard-type solitons α and β, the construction of two-soliton solutions is straightforward and leads to condition (17) on a suitable manifold.

The cases where a γ-type soliton is involved are more interesting. Let us combine one α-type soliton with a γ-type soliton. The dispersion manifold in this case is $(A(p_1) = 0) \cap (A(p_2) = 0) \cap (B(p_2) = 0)$ or $(A(p_1) = 0) \cap (B(p_1) = 0) \cap (A(p_2) = 0)$. Let us choose the first case and write the solution as

$$\begin{aligned} F &= c + ce^{\eta_1} + Ce^{\eta_2} + Ke^{\eta_1+\eta_2} \\ G &= s - se^{\eta_1} + Se^{\eta_2} + Le^{\eta_1+\eta_2}. \end{aligned} \tag{20}$$

We readily find that $K = CM$, $L = -SM$, with the following three possibilities for C, S and M, namely:

(i) Type γ_1: $C = c$, $S = s$ and

$$M = -\frac{A(p_1 - p_2)}{A(p_1 + p_2)}.$$

(ii) Type γ_2: $C = s$, $S = -c$ and

$$M = \frac{B(p_1 - p_2)}{B(p_1 + p_2)}.$$

(iii) Type γ_3: C and S are free and M is given by (16) and therefore (17) must be satisfied again on the appropriate dispersion manifold.
Similar conclusions can be reached in the $\beta + \gamma$ case, *mutatis mutandis*.

Let us finally consider the interaction of two γ-type solitons. Here, the dispersion manifold is $(A(p_1) = 0) \cap (B(p_1) = 0) \cap (A(p_2) = 0) \cap (B(p_2) = 0)$. The two-soliton solution can be written as

$$\begin{aligned} F &= c + C_1 e^{\eta_1} + C_2 e^{\eta_2} + K e^{\eta_1+\eta_2} \\ G &= s + S_1 e^{\eta_1} + S_2 e^{\eta_2} + L e^{\eta_1+\eta_2}. \end{aligned} \tag{21}$$

For notational convenience, we introduce the quantities φ and ψ through

$$K = s\varphi + c\psi \qquad L = c\varphi - s\psi \tag{22}$$

and readily find

$$\begin{aligned} \varphi &= -(C_1 S_2 + C_2 S_1)\frac{B(p_1 - p_2)}{B(p_1 + p_2)} \\ \psi &= -(C_1 C_2 - S_1 S_2)\frac{A(p_1 - p_2)}{A(p_1 + p_2)} \end{aligned} \tag{23}$$

provided that $A(p_1 + p_2) \neq 0$ and $B(p_1 + p_2) \neq 0$.

However, if the two γ-type solitons are obtained for $A = UV$, $B = UW$ through $U(p_1) = U(p_2) = 0$, and, moreover, if U is linear, then both $U(p_1 \pm p_2) = 0$ and $A(p_1 \pm p_2) = B(p_1 \pm p_2) = 0$ as well. In that case, with F and G given by (21), one has $A(F \cdot F - G \cdot G) = BF \cdot G = 0$ for arbitrary C_i, S_i, K and L, and φ and ψ are still totally free at this stage.

As we have seen above, the quantities C_i and S_i ($i = 1, 2$) are, in general, fixed by the interaction of γ-type solitons with α- and β-type solitons, except in the special case of γ_3 where (17) would also be satisfied on the appropriate dispersion manifold. Thus, if $A(p_1+p_2) \neq 0$ and $B(p_1+p_2) \neq 0$ then the only possible values of C_i, S_i, and consequently φ and ψ, are as follows.

(a) $\gamma_1 + \gamma_1$: $C_1 = C_2 = c$, $S_1 = S_2 = s$ and

$$\varphi = -2sc\frac{B(p_1 - p_2)}{B(p_1 + p_2)} \qquad \psi = (s^2 - c^2)\frac{A(p_1 - p_2)}{A(p_1 + p_2)}.$$

(b) $\gamma_1 + \gamma_2$: $C_1 = c$, $C_2 = s$, $S_1 = s$, $S_2 = -c$ and

$$\varphi = (c^2 - s^2)\frac{B(p_1 - p_2)}{B(p_1 + p_2)} \qquad \psi = -2sc\frac{A(p_1 - p_2)}{A(p_1 + p_2)}.$$

(c) $\gamma_2+\gamma_2$: $C_1=C_2=s$, $S_1=S_2=-c$ and

$$\varphi=2sc\frac{B(p_1-p_2)}{B(p_1+p_2)} \qquad \psi=(c^2-s^2)\frac{A(p_1-p_2)}{A(p_1+p_2)}.$$

In the exceptional case (iii) where C and S are free, the γ-type soliton would remain completely free even at this stage.

As the three- or more-soliton solutions exist only for integrable systems, we cannot continue the analysis for general A and B polynomials. The only known integrable cases for which γ-type solitons exist at all do factorize with a linear U; therefore, only the study of the three-soliton solutions could determine the quantities φ and ψ which remain free at the two-soliton level. This staggered determination of the N-soliton solutions from the study of the $(N+1)$-soliton solution was first noted in our work [4] on 'static' solitons, i.e. precisely the case where the common factor U is just D_t.

4. Is there a free γ-type soliton for integrable systems?

The fact that the parameters of a γ-type soliton may remain undetermined even at the level of two-soliton solutions raises questions about the nature of such a solution. That is, should we call this solution a 'soliton' or not. The point is that it is not generally true that any values of C and S in (11) define a *soliton*. For instance, let us assume that the common factor of A and B is just $U=D_t$. Then, since U is a factor of both A and B it is easy to convince oneself that any time-independent F and G will satisfy the equations.

However, not every time-independent object can be called a time-independent soliton: there is the additional requirement that upon interaction with any moving object it should re-emerge unchanged, maybe with a shift in position. The condition for the γ-type object (defined above) to satisfy this criterion is that it reduces to γ_1 ($C=c$ and $S=s$) or γ_2 ($C=s$ and $S=-c$), unless condition (17) happens to be satisfied on the appropriate dispersion manifold, in which case the γ-type soliton is still free at this stage. Now we should note that, while (17) is satisfied on the $\alpha+\alpha$, $\alpha+\beta$ and $\beta+\beta$ manifolds for all the known integrable equations, it is not satisfied on the $\alpha+\gamma$ or $\beta+\gamma$ manifolds for those few known equations where γ-type solitons exist. This means that the γ-type solitons are *in fact fixed at this stage for the integrable cases.*

From an analysis of the conditions for the existence of the two-soliton solutions we obtained condition (17) on various dispersion manifolds depending on the soliton solutions under consideration. This condition is a very strong one. To start with, it constitutes a first necessary condition for integrability. Thus, whenever the partial differential equation (PDE) under consideration has a Hirota form (7) that possesses a continuous vacuum (i.e. $A(0)=B(0)=0$), condition (17) may serve as a first check for the integrability of the equation.

To illustrate this, let us consider the even–even bilinear equation with Hirota polynomials

$$A=D_x^3D_t+D_yD_t+aD_x^2 \qquad B=D_x(D_t+bD_x). \tag{24}$$

If we impose condition (17) on $A(p_1)=A(p_2)=0$, we find the necessary condition $a=b=0$. The same condition is obtained on the manifold $A(p_1)=0$, $k_2=0$ (from the first factor of B). On the other hand, if one wants to satisfy $B(p_2)=0$ through the second factor, i.e. $p_2+bk_2=0$, then (17) is never satisfied, even for $a=b=0$. In this last special case, and in this case only, however, (17) is *not* needed for integrability, as for $p_2=0$,

$a = b = 0$ both A and B vanish and we have, in fact, a γ_1- or γ_2-type soliton. Finally, $a = b = 0$ is a necessary and sufficient condition for the existence of two-soliton solutions of all possible types (α, β, γ_1, γ_2). We have thus quickly obtained the only integrable subcase of (24) at the two-soliton level. If we use instead only the standard-type vacuum, we have to go to three-soliton solutions (and, in exceptional cases, even to four-soliton solutions) in order to restrict the values of a and b.

Conversely, we could use (17) in order to derive the possible forms of the bilinear PDEs which would possess a two-soliton solution in the presence of a continuous vacuum. However, deriving the general solution of the functional equation (17) under the constraints defining the dispersion manifold seems to be a formidable task (in particular, because for some special values of the parameters the existence of a solution is not determined by (17), but by the β-soliton being reduced to a γ-type soliton, as in the case of (24) above). Moreover, in [5] we have used general arguments based on singularity analysis and derived all the possible bilinear even–even PDEs that could be candidates for integrable equations and, in the light of these results, finding the general solution of (17) does not seem to be necessary. In conclusion, (24) with $a = b = 0$ is the only integrable pair with a continuous vacuum (and $A \neq B$).

Let us rewrite the integrable case $a = b = 0$ of (24) in nonlinear form. We first make a $\pi/4$ rotation in F, G space ($F = f + g$, $G = f - g$) and obtain

$$\begin{cases} D_t(D_x^3 + D_y) f \cdot g = 0 \\ D_t D_x (f \cdot f - g \cdot g) = 0. \end{cases} \tag{25}$$

Here, if $f = g$, then the equation reduces to Ito's equation for shallow water waves [6]; the present equation is therefore its generalization. If we put $f = \mathrm{e}^{v} \cos u$, $g = \mathrm{e}^{v} \sin u$, we obtain

$$\begin{cases} u_{xxxt} + 6u_{xt}v_{xx} + u_{yt} = -[v_{xxxt} + 6v_{xx}v_{xt} + v_{yt}] \tan(2u) \\ v_{xt} = u_{xt} \tan(2u). \end{cases} \tag{26}$$

The Ito equation is the term in square brackets involving v. Note also that the second equation is the same as in the sine–Gordon equation (8). Our new equations (26) then represent the coupling of the Ito field v to another field u. The coupling is similar to that of the sine–Gordon, but in this case we cannot eliminate either one of the fields, and we thus have a genuine two-component system.

5. The physical vacuum and the solitons interpolating between them

Let us now return to the question of the vacuum angle. We have already noted that, even when ϕ is fixed, there are, in fact, several 'physical' vacua, and that the soliton solutions connect them. The evidence of this is obtained when we recall that bilinear equations are invariant under a simultaneous change of phase. Thus, (11) can also be written as

$$\begin{aligned} \tilde{F} &:= \mathrm{e}^{-\eta} F = \mathrm{e}^{-\eta} c + C \\ \tilde{G} &:= \mathrm{e}^{-\eta} G = \mathrm{e}^{-\eta} s + S \end{aligned} \tag{27}$$

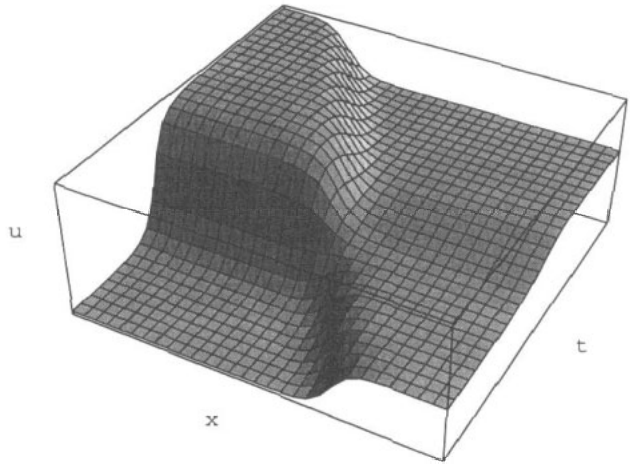

Figure 1. The time evolution of the $\alpha+\beta$-soliton solution of (24) with $a=b=0$ (the integrable case). The four different vacuum levels are clearly visible. (For this equation the β-soliton is x-independent.)

so that the soliton seems to be built on top of the vacuum $\tilde{F}=C$, $\tilde{G}=S$.

Further information on the vacuum is obtained when we use the physical quantities. It is well known that F or G alone do not have physical meaning as they 'blow up' when $\eta\to\infty$. The typical physical variable is

$$u=\tan^{-1}(G/F) \tag{28}$$

so let us see how the vacua look from the point of view of u. The starting vacuum (10) yields $u=\phi$, and this value is also obtained from (11) when $\eta\to-\infty$. When $\eta\to\infty$ we find that the limiting value is different for different solitons.

$$\alpha:\phi\to-\phi \qquad \beta:\phi\to(\pi/2)-\phi$$

$$\gamma_1:\phi\to\phi \qquad \gamma_2:\phi\to\phi-(\pi/2).$$

The solitons do, therefore, connect different values of the vacuum. Note that, from the point of view of u, the vacuum angle is defined only by the modulo π.

When we look at the two-soliton solutions exhibited in section 3, we observe that it is really this change in the angle that characterizes the soliton. For example, (19) connects vacuum angle ϕ with $\phi-(\pi/2)$ and the intermediate vacuum angle is $-\phi$ or $(\pi/2)-\phi$, depending on which order the α and β kinks appear (see figure 1). That is, the values of the soliton's left-hand side and right-hand side vacuum angles may change during the interaction but the operation made on the vacuum angle will stay invariant and may be associated with the soliton. Figure 2 shows how the α-soliton interpolates between different vacua.

Soliton γ_1 is quite curious, because in isolated it goes over to the vacuum. It is therefore a kind of 'ghost' soliton and invisible when on its own. However, when it interacts with a normal soliton it manifests itself in an unambiguous way. Figure 3 shows the time evolution of an $\alpha+\gamma_1$ pair. In figure 3(a) we display the 'physical' field u. It describes a kink that propagates from left to right, and, as expected, the 'ghost' soliton is invisible but its effect

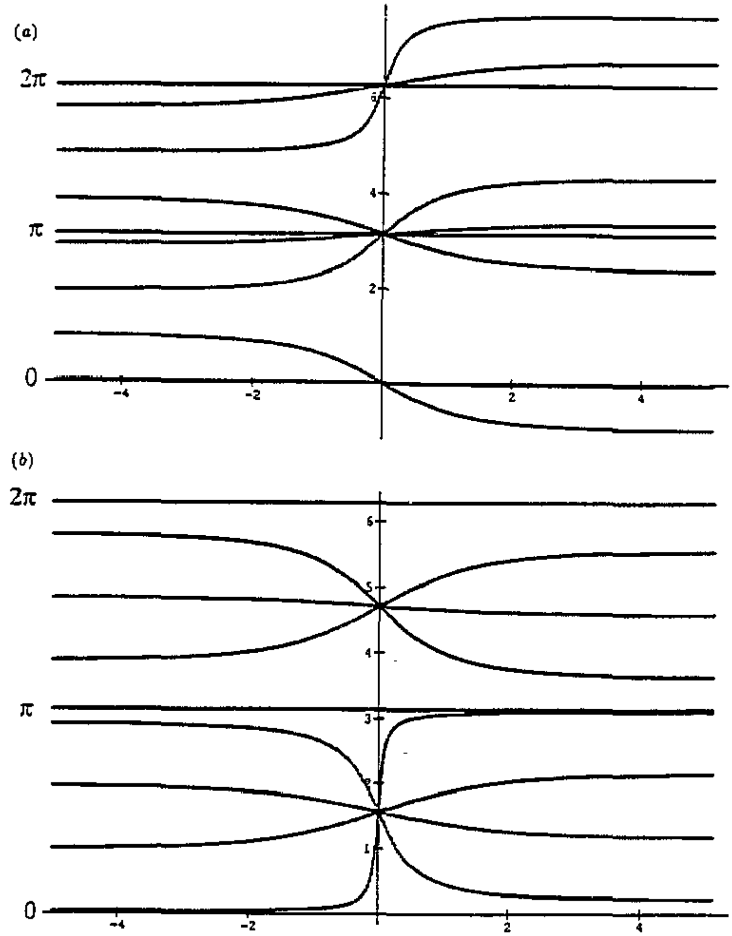

Figure 2. This figure shows how the α-soliton (13) interpolates different vacua. For figure (a) we have assumed that e^{η} in (13) is positive, for figure (b) that it is negative.

is reflected in the phase shift of α. In figure 3(b), we display the field v showing both plane fields and their interaction. The situation is reminiscent of dromions where background ghost-like fields exist. They are also invisible in the physical variable but the effect of their interaction is clearly visible and creates local disturbances [7].

6. Conclusions

To conclude, we remark that the (continuous) vacuum multiplicity is an interesting property of even–even bilinear equations which allows surprising phenomena to occur. One of the most remarkable is the existence of hidden solitons which appear only in interaction with 'normal' solitons. For these systems, it turns out already that the existence of general two-soliton solutions can be used for investigating the integrability in a simple manner. Unfortunately, integrable systems having this property are quite rare and few examples are known to date.

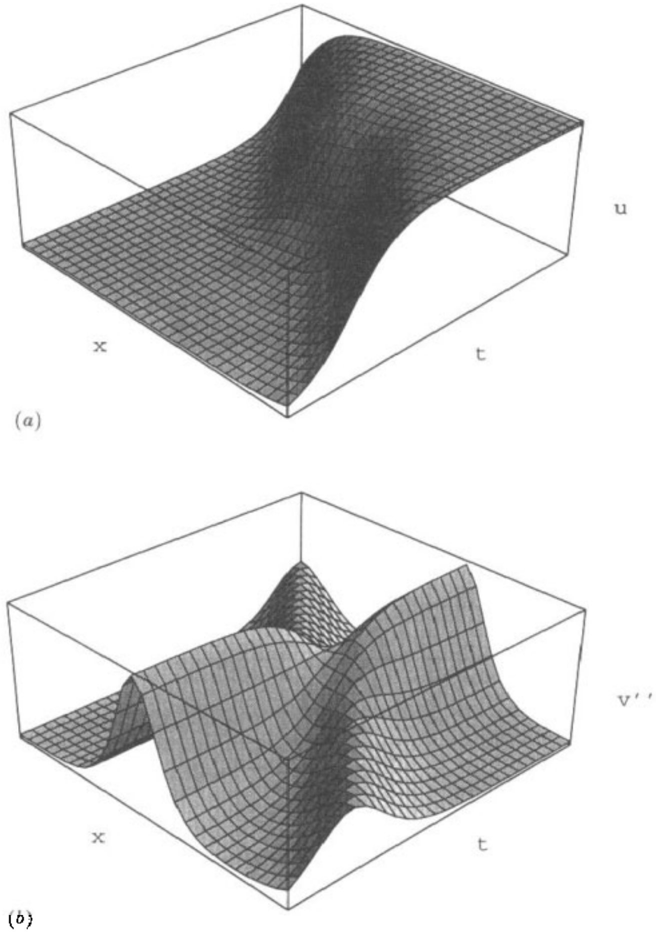

Figure 3. Time evolution of the $\alpha + \gamma_1$ soliton in terms of the variables u in (a) and v in (b). For $t \to \pm\infty$, only the α soliton is visible in the u variable; the γ_1 soliton manifests itself only at the point where the α soliton goes over it. For variable v, both solitons are visible.

Acknowledgment

We would like to thank J Satsuma for comments on an early version of the manuscript.

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